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MATHEMATICS

Graphs of Degree <24, which are Limit Graphs for Minimal Vertex-Primitive Graphs of Type HA[1]

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Throughout the paper, by a graph we mean an undirected graph without loops or multiple edges. The vertex set and edge set of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$. By an automorphism of graph Γ we mean a permutation on the set $V(\Gamma)$ preserving the adjacency relation.

A graph is called a vertex-primitive graph if it admits a primitive on its vertex set group of automorphisms. We denote the class of connected vertex-primitive graphs by $\mathcal{FP}$ (here and below by a class of some graphs we mean a set of isomorphic types of these graphs). Each primitive permutation group can be realized as an edge-transitive automorphism group of some connected graph. Here the most natural realizations are obtained via graphs of minimal degree. A connected graph is called a *graph of minimal* degree for a primitive permutation group of automorphisms G on the set V if it has a minimal degree among all connected graphs Γ, with $V(\Gamma) = V$ and $G \leq \mathrm{Aut}(\Gamma)$. The subclass of the class $\mathcal{FP}$ consisting of all graphs of minimal degree for finite primitive permutation groups is denoted by $\mathcal{FP}^{\min}$.

For investigation of $\mathcal{FP}$ class, it was proposed an approach in [1, 3] connected with investigation of limit graphs for $\mathcal{FP}$ class. If C is an arbitrary class of connected vertex-primitive graphs, then an infinite connected graph, each ball of which is isomorphic to a ball of some graph from C, is called a *limit graph* for C. The class of limit graphs for C is denoted by $\lim(C)$. A description of $\lim(C)$ provides a useful description of the possible local structures of generic graphs from C.

The problem of description of graphs from $\lim(\mathcal{FP})$ was posed by Trofimov in [2]. Mentioned above implies that investigation of the structure of graphs from $\lim(\mathcal{FP}^{\min})$ is also of interest.

The systematic study of $\lim(\mathcal{FP})$ class was started by Giudici et al. in [3]. In this connection it was shown that

$$\lim(\mathcal{FP}) = \lim(\mathcal{FP}_{\mathrm{HA}}) \cup \lim(\mathcal{FP}_{\mathrm{AS}}) \cup \lim(\mathcal{FP}_{\mathrm{PA}}),$$

where the subclasses $\mathcal{FP}_{\mathrm{HA}}$, $\mathcal{FP}_{\mathrm{AS}}$, $\mathcal{FP}_{\mathrm{PA}}$ of class $\mathcal{FP}$ are defined as follows. Let G be a primitive permutation group acting on the set V and $v \in V$. If G has an abelian normal regular subgroup, then G is said to be of type HA. If G is an almost simple group, i.e., there exists a finite nonabelian simple group T such that $\mathrm{Inn}(T) \trianglelefteq G \leq \mathrm{Aut}(T)$, then G is said to be of type AS. If G has a minimal normal subgroup $N = T^k$ $(k \geq 2)$ for some nonabelian simple group T and the stabiliser in N of v is nontrivial and has no composition factor isomorphic to T, then G is said to be of type PA. For each primitive type X, the class of all vertex-primitive graphs with an automorphism group of type X is denoted by $\mathcal{FP}_{\mathrm{X}}$, and the subclass of $\mathcal{FP}_{\mathrm{X}}^{\min}$ consisting of all graphs of minimal degree for vertex-primitive groups of type X is denoted by $\mathcal{FP}_{\mathrm{X}}$.

It follows from [3, Theorem 1] that

$$\lim(\mathcal{FP}^{\min}) = \lim(\mathcal{FP}_{\mathrm{HA}}^{\min}) \cup \lim(\mathcal{FP}_{\mathrm{AS}}^{\min}) \cup \lim(\mathcal{FP}_{\mathrm{PA}}^{\min}).$$

Therefore the investigation of $\lim(\mathcal{FP}_{\mathrm{HA}}^{\min})$ class, which is of independent interest, is a necessary stage of the investigation of $\lim(\mathcal{FP}^{\min})$ class.

In [4–7], all graphs of degree ≤ 14 from $\lim(\mathcal{FP}_{\mathrm{HA}}^{\min})$ (12 graphs) were found, and also a countable set of setwise-nonisomophic graphs of degree 24 from

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$\lim(\mathcal{FP}^{\min}_{\mathrm{HA}})$ was found. In the present paper, we find all graphs of degree <24 from $\lim(\mathcal{FP}^{\min}_{\mathrm{HA}})$ (23 graphs, see Theorem 1 below).

Let d be a positive integer. Let M be a set of generators of $\mathbb{Z}^d$ such that $M = -M$ and $0 \notin M$. Recall that a graph $\mathbb{Z}^d$, M with vertex set $\mathbb{Z}^d$ and the set of edges such that two vertexes are connected by an edge iff their difference lies in $M = -M$ and $0 \notin M$ is called a Caley graph of the group $\mathbb{Z}^d$ corresponding to the set of generators M. Let $\mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$ be a stabilizer of vertex 0 in automorphism group of $\Gamma_{\mathbb{Z}^d,M}$. We say a graph $\Gamma_{\mathbb{Z}^d,M}$ to be minimal Caley graph of $\mathbb{Z}^d$ if M is an orbit of minimal cardinality of group $\mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$, acting on the set $\mathbb{Z}^d\backslash\{0\}$. The class of all Cayley graphs of the group $\mathbb{Z}^d$ is denoted by $\mathrm{Cay}(\mathbb{Z}^d)$, and the class of all minimal Caley graphs of $\mathbb{Z}^d$ is denoted by $\mathrm{Cay}^{\min}(\mathbb{Z}^d)$. It follows from [3, Theorem 2] that each element of $\lim(\mathcal{FP}_{\mathrm{HA}})$ lies in $\mathrm{Cay}(\mathbb{Z}^d)$ for some d. Morever, it follows from [3, Theorem 2] and the definition of a limit graphs that each element of $\lim(\mathcal{FP}^{\min}_{\mathrm{HA}})$ lies in $\mathrm{Cay}^{\min}(\mathbb{Z}^d)$ for some d. In other words, $\lim(\mathcal{FP}^{\min}_{\mathrm{HA}}) \subseteq \bigcup_{d=1}^{\infty}\mathrm{Cay}^{\min}(\mathbb{Z}^d)$. For each $i \in \mathbb{Z}_{\geq 0}$, we identify $\mathbb{Z}^d$ with a set of integer row vectors of length d with coordinatewise addition. For $i \in \{1, 2, \ldots, d\}$, let e_i be a row-vector of length d having 1 at position i and 0 at other positions. Next we set $M_{d,1} = \{\pm e_i\colon i \in \{1, 2, \ldots, d\}\}$ and

$$M_{d,2} = M_{d,1} \cup \left\{\pm\sum_{i=1}^{d} e_i\right\}.$$

Theorem 1. *The class of all graphs from* $\lim(\mathcal{FP}^{\min}_{\mathrm{HA}})$ *of degree <24 is equal to the class of all graphs from* $\bigcup_{d=1}^{\infty}\mathrm{Cay}^{\min}(\mathbb{Z}^d)$ *of degree <24 and consists of following graphs:*

$\Gamma_{\mathbb{Z},M_{1,1}}$ *of degree* 2;

$\Gamma_{\mathbb{Z}^2,M_{2,1}}$ *of degree* 4;

$\Gamma_{\mathbb{Z}^2,M_{2,2}}$ *and* $\Gamma_{\mathbb{Z}^3,M_{3,1}}$ *of degree* 6;

$\Gamma_{\mathbb{Z}^4,M_{4,1}}$ *of degree* 8;

$\Gamma_{\mathbb{Z}^4,M_{4,2}}$ *and* $\Gamma_{\mathbb{Z}^5,M_{5,1}}$ *of degree* 10;

$\Gamma_{\mathbb{Z}^4,M_{4,3}}$, $\Gamma_{\mathbb{Z}^5,M_{5,2}}$, *and* $\Gamma_{\mathbb{Z}^6,M_{6,1}}$ *of degree* 12, *where* $M_{4,3} = M_{4,1} \cup \pm\{e_1 + e_2 + e_3 + e_4\}$;

$\Gamma_{\mathbb{Z}^6,M_{6,2}}$ *and* $\Gamma_{\mathbb{Z}^7,M_{7,1}}$ *of degree* 14;

$\Gamma_{\mathbb{Z}^7,M_{7,2}}$ *and* $\Gamma_{\mathbb{Z}^8,M_{8,1}}$ *of degree* 16;

$\Gamma_{\mathbb{Z}^6,M_{6,3}}$, $\Gamma_{\mathbb{Z}^8,M_{8,2}}$, *and* $\Gamma_{\mathbb{Z}^9,M_{9,1}}$ *of degree* 18, *where* $M_{6,3} = M_{6,1} \cup \pm\{e_1 + e_2, e_3 + e_4, e_5 + e_6\}$;

$\Gamma_{\mathbb{Z}^6,M_{6,4}}$, $\Gamma_{\mathbb{Z}^8,M_{8,3}}$, $\Gamma_{\mathbb{Z}^9,M_{9,2}}$, *and* $\Gamma_{\mathbb{Z}^{10},M_{10,1}}$ *of degree* 20, *where* $M_{6,4} = M_{6,1} \cup \pm\{e_1 - e_2 + e_4, e_1 - e_3 + e_5, e_2 - e_3 + e_6, e_4 - e_5 + e_6\}$,

$M_{8,3} = M_{8,1} \cup \pm\{e_1 + e_2 + e_3 + e_4, e_5 + e_6 + e_7 + e_8\}$;

$\Gamma_{\mathbb{Z}^{10},M_{10,2}}$ *and* $\Gamma_{\mathbb{Z}^{11},M_{11,1}}$ *of degree* 22.

For each graph $\Gamma_{\mathbb{Z}^d,M}$, the group $\mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$ is a subgroup of $\mathrm{GL}_d(\mathbb{Z})$, and acts naturally on $\mathbb{Z}^d = V(\Gamma_{\mathbb{Z}^d,M})$. Besides that, the group $\mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$ induces a permutation group on the set $\bar{M}$ of pairs of mutually inverse vectors from the set M. The conditions of Theorem 1 imply $|\bar{M}| \leq 12$. To prove Theorem 1, for each minimal transitive permutation group S of degree ≤12, we find all systems generator systems M of $\mathbb{Z}^d$, which are G-orbits on $\mathbb{Z}^d\backslash\{0\}$ of minimal cardinality for some subgroup G of $\mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$, such that G induces on $\bar{M}$ a permutationaly isomorphic to S group. Herewith we search for systems M up to the following equivalency, which implies isomorphism of the corresponding Caley graphs. Two systems of generators M_1 and M_2 of a group $\mathbb{Z}^d$ are equivalent if $M_2 = M_1A$ for some $A \in \mathrm{GL}_d(\mathbb{Q})$. As a result, we get all graphs from $\bigcup_{d=1}^{\infty}\mathrm{Cay}^{\min}(\mathbb{Z}^d)$ of degree <24. To prove that each found in this way graph $\Gamma_{\mathbb{Z}^d,M}$ is contained in $\lim(\mathcal{FP}^{\min}_{\mathrm{HA}})$, we construct an infinity set of graphs $\Gamma_{\mathbb{Z}^d_{p_i},\varphi_{p_i}(M)}$, $i \in \mathbb{Z}_{\geq 0}$ from the class $\mathcal{FP}^{\min}_{\mathrm{HA}}$, where p_i is a prime number, $p_{i+1} > p_i$, $\mathbb{Z}^d_{p_i}$ is a d-dimensional space of the residues modulo p_i, and $\varphi_{p_i}\colon \mathbb{Z}^d \to \mathbb{Z}^d_{p_i}$ is a function that substitutes integer elements of row vectors my their residues modulo p_i. In Theorem 1 proof, we use the classification of maximal finite subgroups of group $\mathrm{GL}_d(\mathbb{Z})$ for $d \leq 11$ and the classification of transitive permutation groups for small degrees (both classifications are available in [8]). Also we use earlier results from [4–7] about the structure of class $\lim(\mathcal{FP}^{\min}_{\mathrm{HA}}) \cap \mathrm{Cay}^{\min}(\mathbb{Z}^d)$ for $d \leq 7$.

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